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The Embedding of Unstable Non-relativistic Particles into Galilean Quantum Field Theories

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Abstract

Some unpleasant features of the usual treatment of irreversible processes in quantum mechanics are discussed. It is shown how the description of a non-relativistic unstable particle can be cleanly embedded into a reversible Galilean quantum field theory. It is proven that in the case of stable particles the embedding procedure gives the same values for internal energy which are obtained by the usual procedure. Finally the technique is applied to the 'Galilee' model.

1. *Introduction*

In this paper we are concerned with the problem of describing irreversible processes in the framework of quantum mechanics. Such a problem is met at the macroscopic as well as at the microscopic level.

(a) A basic point for physics is that we can give to the macroscopic observables of macroscopic systems objective (i.e. independent of any observation) values and that the time evolution of such values is given by self-contained, irreversible equations of motion (e.g. the Boltzmann equation for a dilute gas, the Navier–Stokes equations for the hydrodynamical stage, etc.). Such points have a fundamental relevance for quantum mechanics itself: indeed the 'realistic' interpretation of quantum mechanics requires the existence of a macroscopic 'classical' level of description (Ludwig, 1953, 1954, 1955; Daneri *et al.*, 1962, 1966; Prosperi, 1967; Lanz *et al.*, 1971; Rosenfeld, 1965). Therefore for reasons of coherence such a prerequisite must in turn be deduced from the quantum mechanical treatment of N-body systems. A very significant step in such a deduction is the so-called 'master equation' (Pauli, 1928; Van Hove, 1955; Prigogine, 1962), which is a linear and irreversible equation for the time evolution of the relevant observables. Unfortunately the master equation cannot be

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deduced in an exact way from the basic reversible equation of motion for the N-body system. One must be content with an approximate result; sufficient conditions for the reliability of the involved approximations can be given, but it appears hard to verify such conditions in realistic cases (Prigogine, 1962; Van Hove, 1957; Zwanzig, 1961; Lanz & Ramella, 1969; Lanz & Lugiato, 1970).

(b) Some of the known unstable particles have a lifetime long enough to be experimentally detectable, and their interpretation as particles is obvious. As for the very short-lived ones, the so-called resonances, strong arguments exist supporting their interpretation as particles. Typically, in the theory of unitary symmetries, stable particles and resonances can be put in the same multiplet. However, stable and unstable particles are not treated on the same footing in quantum field theory. In fact stable particles are defined by association with an irreducible component of the unitary representation of the Poincaré group given by the field theory; this definition cannot be extended to unstable particles, due to the unitarity of time translations in field theory. Further, we know experimentally that the long-lived particles decay to a high degree of accuracy by an exponential law. However, in field theory there is no state vector leading in an exact way to an exponential decay; there are only states giving an exponential decay within a certain approximation (Levy, 1959; Höhler, 1958; Araki *et al.*, 1957; Glaser & Källen, 1956–57; Zumino, 1961). In any case there is a large arbitrariness in the choice of such states.

The origin of these difficulties is, in our opinion, linked to the following facts. A physical object is characterised in quite a natural way by a suitable set of observables; e.g. a macroscopic object by a set of macroscopic observables, a particle by its position, momentum and spin. However, we usually associate to such objects a Hilbert space $\mathscr{H}$ in which many more variables can be represented; these variables describe a fundamental ideal structure which is supposed to be underlying to the objects we are considering. Due to the hypothetical character of this structure, such variables need not be observable; examples thereof are the N-particle structure of all macroscopic systems and the field structure of particles. In general we shall call the ideal structure underlying to an object S the 'model' of S. The sets of compatible observables of S are not at all complete systems of operators in the Hilbert space $\mathscr{H}$ of the model, so that one does not know how to associate a state vector to the object. Usually such a difficulty is bypassed by *ad hoc* assumptions; a typical example is the equiprobability and random phase assumptions to select the initial statistical operator for a macroscopic system. Besides this intrinsic ambiguity, self-contained dynamics for the typical observables of the object is obtained at the cost of rather questionable approximations. On the other hand, it seems to us that for the operational individuation of an object the existence of a statistical causality for its observables plays a fundamental role. Therefore the coexistence of self-contained dynamical descriptions for the objects with the model describing their structure is, in our opinion, a crucial problem. The aim of this paper is to show in a particularly simple situation that introducing a suitable mathematical procedure of embedding of the object into the model such a coexistence becomes straightforward. This procedure is in a sense the particle-field analogous of the "embedding" recently proposed by Ludwig to exhibit the coexistence of the macro-dynamics with the N-body theory (Ludwig, 1972, p. 116 ff).

In the present paper we consider the embedding problem for a non-relativistic (generally unstable) particle into a Galilean field-theoretical model (Levy-Leblond, 1967). For such an object the self-contained dynamics is completely characterised by the symmetry under transformations of the Galilei group and the requirement that a position operator exists, as has been discussed in a previous paper (Lanz *et al.*, 1973). The embedding procedure essentially fixes the numerical values of spin, lifetime and internal energy. The complete description of the decay process, including the decay products, will be given in a forthcoming paper.

In Section 2 we recall briefly and complete the results of Lanz *et al.* (1973) on the description of a non-relativistic particle. In Section 3 we characterise the class of models we consider; in Section 4 we define the embedding procedure, which is concretised in Section 5 into a suitable technique. We prove further that in the case of a stable particle such an embedding technique provides the same results as the usual procedure. The technique is finally applied to the 'Galilee' model (Levy-Leblond, 1967) in Section 6.

2. *Description of a Non-Relativistic Particle*

Let us consider a particle O with mass M and spin j. According to the results of Lanz *et al.* (1973), which we recall briefly, its description is given in the space $\mathscr{H} \times R$ of couples (f, t), where $-\infty < t < +\infty$ and f belongs to a Hilbert space $\mathscr{H}$. $\mathscr{H}$ is the direct integral of Hilbert spaces $\mathfrak{H}(\mathbf{p})$, $\mathbf{p} \in R_3$, with the Lebesgue measure, $\mathscr{H} = \int^{\oplus} \mathfrak{H}(\mathbf{p})\, d_3\mathbf{p}$, $\mathfrak{H}(\mathbf{p})$ being the space $\mathfrak{H}^{(j)}$ of the irreducible representation $\mathscr{D}^{(j)}$ of $SU(2)$. $\mathscr{H}$ is the space of a projective, unitary, irreducible representation of the group $\mathscr{G}_0$ of space translations, rotations and accelerations:

$$T(\mathbf{a})\{f(\mathbf{p})\} = \{e^{-i\mathbf{a}\mathbf{p}} f(\mathbf{p})\} \tag{2.1a}$$

$$O(R)\{f(\mathbf{p})\} = \{D^{(j)}(R) f(R^{-1}\mathbf{p})\} \tag{2.1b}$$

$$G(\mathbf{v})\{f(\mathbf{p})\} = \{f(\mathbf{p} - M\mathbf{v})\} \tag{2.1c}$$

where $\{f(\mathbf{p})\} \in \mathscr{H}$, $\mathbf{a} \in R_3$, $R \in SU(2)$, $\mathbf{v} \in R_3$; $T(\mathbf{a})$, $O(R)$ and $G(\mathbf{v})$ are the operators representing space translations, rotations and accelerations respectively; $D^{(j)}(R)$ is the operator representing R in $\mathscr{D}^{(j)}$.

Any 'maximal sharp' property γ of O is placed in correspondence to an element $f_\gamma \in \mathscr{H}$. To an observation of the 'maximal sharp' property γ of O performed at time t the couple (f_γ, t) corresponds. If a collection of N particles has by preparation at time t_0 the property γ_0, the number $N_{\gamma_1 t_1}$

of particles for which the property γ_1 is ascertained at a time $t_1 \geqslant t_0$ is given by

$$N_{\gamma_1, t_1} = |(f_{\gamma_1}, V(t_1 - t_0) f_{\gamma_0})|^2 . N \tag{2.2}$$

where $V(t)$, $t \geqslant 0$, is a semigroup of contractive linear operators given by

$$V(t)\{f(\mathbf{p})\} = \left\{\exp\left[-i\left(\frac{p^2}{2M} + \lambda\right)t\right]f(\mathbf{p})\right\}, \qquad t \geqslant 0 \tag{2.3}$$

where $\lambda = U - i(\gamma/2)$, U being the internal energy of the particle and γ the inverse lifetime. The symmetry of the theory under the transformations of the proper orthochronous Galilei group $\mathscr{G}$ $g \equiv (b, \mathbf{a}, \mathbf{v}, R)$ is expressed by the fact that $\mathscr{H} \times \mathscr{R}$ is the space of a representation of $\mathscr{G}$ given by

$$\mathscr{U}(b, \mathbf{a}, \mathbf{v}, R)(f, t) = (U(\mathbf{a} + \mathbf{v}t, \mathbf{v}, R) f, t + b) \tag{2.4}$$

with

$$U(\mathbf{a}, \mathbf{v}, R) = T(\mathbf{a})\, G(\mathbf{v})\, O(R) \tag{2.5}$$

and

$$\begin{aligned} &|(U(\mathbf{a} + \mathbf{v}t_1, \mathbf{v}, R) f, V(t_1 - t_0)\, U(\mathbf{a} + \mathbf{v}t_0, \mathbf{v}, R) g)| \\ &\quad = |(f, V(t_1 - t_0) g)|, \qquad f, g \in \mathscr{H} \end{aligned} \tag{2.6}$$

$\mathscr{H} \times \mathscr{R}$ is also the space of a representation of the full Galilei group, such that space inversion and time inversion are given respectively by

$$\mathscr{P}(f, t) = (Pf, t) \tag{2.7}$$

$$\mathscr{T}(f, t) = (Tf, -t) \tag{2.8}$$

with

$$P\{f(\mathbf{p})\} = \{f(-\mathbf{p})\} \qquad \text{or} \qquad P\{f(\mathbf{p})\} = \{-f(-\mathbf{p})\} \tag{2.9}$$

according to the intrinsic parity of the particle, and

$$T\left\{\sum_{m=-j}^{+j} f_m(\mathbf{p})\, u_m\right\} = \left\{\sum_{m=-j}^{+j} f_{-m}^*(-\mathbf{p})\,(-1)^m u_m\right\} \tag{2.10}$$

u_m, $m = -j, -j+1, \ldots, +j$ is the orthonormal set of eigenstates of S_z, with the phases chosen in such a way that

$$S_+ u_{m-1} = \sqrt{[(j+m)(j-m+1)]}\, u_m \tag{2.11}$$

with

$$S_+ = S_x + iS_y$$

where S_x, S_y, S_z are the generators of rotations in $\mathscr{D}^{(j)}$. The set $T(\mathbf{a})$, $V(-t)$, $O(R)$, $G(\mathbf{v})$, P is an irreducible projective, generally non-unitary representation on $\mathscr{H}$ of the orthochronous Galilei semigroup.

3. *Galilean Field Theoretic Models*

The typical models which are supposed to explain the structure of particles are relativistic quantum field theories. Since we have so far developed a Galilean formalism we shall discuss the embedding of non-relativistic particles into a Galilean field theory. Despite its unrealistic character, such a problem has, in our opinion, a formal and methodological relevance. Galilean interacting field theories can be very easily constructed introducing interactions between free particle fields by means of an interaction Hamiltonian which commutes with all the operators representing the Lee algebra of free fields, except the free fields Hamiltonian. Galilean field theories are extensively discussed in Levy-Leblond (1967). We mention here the fact that the Hilbert space of the field theory decomposes into the direct sum of superselection spaces $\mathscr{H}_M$, each one corresponding to a fixed mass M. In each $\mathscr{H}_M$ one has a projective representation of the Galilei group with factors depending on M; the Hilbert space $\mathscr{H}_M$ is spanned by many-particle states, the number of particles of each type being such that the total mass equals M. The dynamics of the fields can describe general processes in which the total mass M is conserved, i.e. not only scatterings but also decay and production processes.

The model we take as underlying to a particle with mass M is the superselection sector with the same mass M of a suitable Galilean field theory. The corresponding Hilbert space $\mathscr{H}_M$ (we shall drop the index M in the sequel) is the space of a projective unitary representation of $\mathscr{G}$; then

$$\mathscr{H} = \int^{\oplus} \bar{h}(\mathbf{p})\, d\mathbf{p} \tag{3.1}$$

where $\bar{h}(\mathbf{p}) = \bar{h}_0$, $\bar{h}_0$ being a suitable Hilbert space and up to an equivalence (Wightman, 1962) one has

$$\bar{T}(\mathbf{a})\{\bar{f}(\mathbf{p})\} = \{e^{-i\mathbf{a}\cdot\mathbf{p}}\bar{f}(\mathbf{p})\} \tag{3.2}$$

$$\bar{U}(t)\{\bar{f}(\mathbf{p})\} = \{e^{i\bar{E}_0 t}\, e^{i(p^2/2M)t}\bar{f}(\mathbf{p})\} \tag{3.3}$$

$$\bar{O}(R)\{\bar{f}(\mathbf{p})\} = \{\bar{D}(R)\bar{f}(R^{-1}\mathbf{p})\} \tag{3.4}$$

$$\bar{G}(\mathbf{v})\{\bar{f}(\mathbf{p})\} = \{\bar{f}(\mathbf{p} - M\mathbf{v})\} \tag{3.5}$$

where $\bar{U}(t) = e^{i\bar{H}t}$ is the operator representing time translations, $\bar{D}(R)$ gives a unitary (in general non-irreducible) representation of $SU(2)$ on $\bar{h}_0$, $e^{i\bar{E}_0 t}$ is a unitary one-parameter group which commutes with $\bar{D}(R)$, $\forall R \in SU(2)$. The meaning of the other symbols is obvious.

We assume that the field theoretic model is invariant under space and time reflections. Then the representation of $\mathscr{G}$ in $\mathscr{H}$ can be extended to a representation of the full Galilei group

$$\bar{P}\{\bar{f}(\mathbf{p})\} = \{\bar{I}_0 f(-\mathbf{p})\} \tag{3.6}$$

$$\bar{T}\{\bar{f}(\mathbf{p})\} = \{\bar{T}_0\bar{f}(-\mathbf{p})\} \tag{3.7}$$

where $\bar{I}_0$ is a unitary and $\bar{T}_0$ an anti-unitary operator on $\bar{h}_0$; both commute with $\bar{E}_0$, $\bar{D}(R)$, $\forall R \in SU(2)$.

We stress that the assumption that the field theoretic model is invariant under space reflection is not necessary for the embedding procedure; if the model is not P-invariant, all the following considerations, except the ones concerning parity, hold unaltered. Also the T-invariance is not strictly necessary; however, to perform the embedding, one needs an anti-unitary symmetry $\bar{T}'$. The simplest case is $\bar{T}' = \bar{T}$, but one might consider also other possibilities, as e.g. $\bar{T}' = \bar{P}\bar{T}$; in such a case $\bar{T}$ should be replaced by $\bar{T}'$ in the following.

4. *Definition of the 'Embedding'*

The particle O is embedded into the model if for any $\varepsilon > 0$ and any finite set τ of non-negative time points two mappings $\mathscr{G}_+$ and $\mathscr{G}_-$ from $\mathscr{H} \times \mathscr{R}$ to $\bar{\mathscr{H}}$, i.e.

$$\overleftarrow{\mathscr{G}}_+(f,t) = \bar{f}_+(t), \qquad \overleftarrow{\mathscr{G}}_-(f,t) = \bar{f}_-(t), \qquad f \in \mathscr{H}, \qquad \bar{f}_\pm(t) \in \bar{\mathscr{H}} \tag{4.1}$$

exist with the following properties:

(a) for any $f_1, f_2 \in \mathscr{H}$

$$\overleftarrow{\mathscr{G}}_\pm(\alpha f_1 + \beta f_2, t) = \alpha \overleftarrow{\mathscr{G}}_\pm(f_1, t) + \beta \overleftarrow{\mathscr{G}}_\pm(f_2, t) \tag{4.2}$$

(b) the mappings $\overleftarrow{\mathscr{G}}_\pm$ are compatible with symmetries

$$\overleftarrow{\mathscr{G}}_\pm \mathscr{U}(g)(f,t) = \omega(g,t)\, \bar{U}(g) \overleftarrow{\mathscr{G}}_\pm(f,t), \qquad g \in \mathscr{G} \tag{4.3}$$

where $\omega(g,t)$ is a suitable phase factor, and

$$\bar{U}(b, \mathbf{a}, \mathbf{v}, R) = \bar{U}(b)\, \bar{T}(\mathbf{a})\, G(\mathbf{v})\, \bar{O}(R) \qquad \overleftarrow{\mathscr{G}}_\pm \mathscr{P} = \bar{P} \overleftarrow{\mathscr{G}}_\pm \tag{4.4}$$

(c) for any $f, g \in \mathscr{H}$, $\|f\|_{\mathscr{H}} = \|g\|_{\mathscr{H}} = 1$, $t - t_0 \in \tau$

$$\left| |(f, V(t-t_0) g)_{\mathscr{H}}|^2 - |(\bar{f}_-(t), g_+(t_0))_{\bar{\mathscr{H}}}|^2 \right| < \varepsilon \tag{4.5}$$

(d) $\overleftarrow{\mathscr{G}}_+$ and $\overleftarrow{\mathscr{G}}_-$ are linked by the relation

$$\overleftarrow{\mathscr{G}}_+ \mathscr{T} = \bar{T} \overleftarrow{\mathscr{G}}_- \tag{4.6}$$

Let us comment on the points (a)–(d). From (4.3) for $g = (t, \mathbf{0}, \mathbf{0}, I)$ we have

$$\bar{f}_\pm(t) = e^{i\bar{H}.t} \bar{f}_\pm(0) \tag{4.7}$$

therefore the mappings $\overleftarrow{\mathscr{G}}_\pm$ can be written more explicitly as

$$\overleftarrow{\mathscr{G}}_\pm(f,t) = e^{i\bar{H}t} \overleftarrow{G}_\pm f \tag{4.8}$$

where by (a) $\overleftarrow{G}_+$ are linear operators from $\mathscr{H}$ to $\bar{\mathscr{H}}$. From (4.3) and (4.4) we have, assuming for simplicity $\omega(g,0) = 1$,

$$\overleftarrow{G}_\pm U(g) = \bar{U}(g) \overleftarrow{G}_\pm, \qquad g \in \mathscr{G}_0 \tag{4.9}$$

$$\omega(g,t) = \exp(-\tfrac{1}{2} M v^2 t)$$

$$\overleftarrow{G}_\pm P = \bar{P} \overleftarrow{G}_\pm \tag{4.9'}$$

Requirement (c) says that all theoretical provisions about the particle given by (2.2) can be obtained using the model as well. In fact

$$\left| |(f_\gamma, V(t_1 - t_0) f_{\gamma_0})_{\mathscr{H}}|^2 - |(e^{i\bar{H}t} \overleftarrow{G}_- f_\gamma, e^{i\bar{H}t_0} \overleftarrow{G}_+ f_{\gamma_0})_{\bar{\mathscr{H}}}|^2 \right| < \varepsilon, \qquad t - t_0 \in \tau \tag{4.10}$$

The element $\exp(i\bar{H}t)\overleftarrow{G}_- f_\gamma$ ($\exp(i\bar{H}t_0)\overleftarrow{G}_+ f_{\gamma_0}$) represents in the model the maximal sharp property $\gamma(\gamma_0)$ observed (prepared) at time $t(t_0)$. The restriction to a finite set of time points and the presence of the arbitrarily small ε are irrelevant from a physical point of view, since in any actual experiment one makes a finite number of measurements affected by a finite error. However from a mathematical point of view the presence of the ε and the restriction to the set τ has the effect that conditions (a)–(d) do not uniquely characterise the operators $\overleftarrow{G}_\pm$. In fact, for given ε and τ there is a class of operators $\{\overleftarrow{G}_+, \overleftarrow{G}_-\}_{\varepsilon,\tau}$; if one takes $\varepsilon' \leqslant \varepsilon$, $\tau' \supseteq \tau$ one has

$$\{\overleftarrow{G}_+, \overleftarrow{G}_-\}_{\varepsilon',\tau'} \subseteq \{\overleftarrow{G}_+, \overleftarrow{G}_-\}_{\varepsilon,\tau}$$

However (4.5) is a strong condition, even if one does not require that the intersection of the $\{\overleftarrow{G}_+, \overleftarrow{G}_-\}_{\varepsilon,\tau}$ for all ε, τ is non-void. In the case that such intersection is non-void, i.e. when at least a couple $\overleftarrow{G}_+$, $\overleftarrow{G}_-$ exists such that

$$|(f, V(t - t_0) g)_{\mathscr{H}}|^2 = |(e^{i\bar{H}t} \overleftarrow{G}_- f, e^{i\bar{H}t_0} \overleftarrow{G}_+ g)_{\bar{\mathscr{H}}}|^2, \qquad \forall t - t_0 \geqslant 0 \tag{4.11}$$

one has a 'strong' embedding relation. As we shall see in the next section, equation (4.11) holds with $\overleftarrow{G}_+ = \overleftarrow{G}_-$ in the case of stable particles. On the contrary, in the case of unstable particles one does not expect that (4.11) can be satisfied. In fact one sees e.g. that (4.11) cannot be satisfied with $\overleftarrow{G}_- = \overleftarrow{G}_+$ due to the fact that the Hamiltonian $\bar{H}$ is bounded from below (see Khalfin, 1957).

The embedding is performed by means of two different operators $\overleftarrow{\mathscr{G}}_+$ and $\overleftarrow{\mathscr{G}}_-$ because the description of the time evolution of an unstable particle is 'one-sided', i.e. a statistical causality exists only if a definite versus of the time evolution is chosen. In fact, $\overleftarrow{\mathscr{G}}_-$ and $\overleftarrow{\mathscr{G}}_+$ embed into the model respectively the couples (f, t), (g, t_0) such that $t \geqslant t_0$. Only in the case of unitary $V(t)$ one requires $\overleftarrow{\mathscr{G}}_- = \overleftarrow{\mathscr{G}}_+$. One-sided evolution is not in contradiction with time inversion invariance; in fact if the versus of the time axes is inverted a description equivalent to the previous one is obtained by the following transformations:

$$\begin{aligned} (f, t) &\to \mathscr{T}(f, t) = (Tf, -t) \stackrel{\text{def}}{=} (f^T, -t) \\ V(t) &\to TV(t)T \stackrel{\text{def}}{=} V(-t), \qquad t \geqslant 0 \end{aligned} \tag{4.12}$$

since

$$|(f_\gamma^T, V(-t + t_0) f_{\gamma_0}^T)_{\mathscr{H}}| = |(f_\gamma, V(t - t_0) f_{\gamma_0})_{\mathscr{H}}| \tag{4.13}$$

Equation (4.5) implies with (4.13) that

$$\left| |(f^T, V(-t+t_0)g^T)_{\mathscr{H}}|^2 - |(\bar{f}_-^T(-t), \bar{g}_+^T(-t_0))_{\bar{\mathscr{H}}}|^2 \right| < \varepsilon$$

$$\bar{f}_\pm^T(t) = (\bar{T}\overleftarrow{\mathscr{G}}_\pm \mathscr{T})(f^T, t) \tag{4.14}$$

We require that the versus of the time axis does not affect the embedding operators so that, taking into account that $-t_0 > -t$, (4.6) follows, from which

$$\overleftarrow{G}_+ T = \bar{T}\overleftarrow{G}_- \tag{4.15}$$

5. *The Embedding Technique*

In the usual approach the stable particles of mass M are associated to the irreducible components of the representation of $\mathscr{G}$ in the superselection sector of the Galilei field theory corresponding to M.

The space of an irreducible component of $\mathscr{G}$ is the subspace of $\bar{\mathscr{H}}$ of the elements $\{\bar{f}(\mathbf{p})\}$ such that

$$\bar{f}(\mathbf{p}) \in \bar{h}^{(J,U)} \tag{5.1}$$

where $\bar{h}^{(J,U)}$ is the space of the representation $\mathscr{D}^{(J)}$ of $SU(2)$ and also the eigenspace of $\bar{E}_0$ corresponding to the eigenvalue U. To such an irreducible component a particle with mass M, internal energy U and spin j corresponds. Let us consider a stable particle characterised by M, U, j and described as in Section 2 and a Galilean field theory, the M superselection sector of which contains an irreducible component of the representation of $\mathscr{G}$ characterised by the same values U and j. Then a 'strong' embedding can be immediately performed with

$$\overleftarrow{G}_+ = \overleftarrow{G}_- = \overleftarrow{G}$$

$$\overleftarrow{G}\left\{\sum_{m=-j}^{+j} f_m(\mathbf{p})\, u_m\right\} = \left\{\sum_{m=-j}^{+j} f_m(\mathbf{p})\, \bar{v}_{m-1}^{(J,U)}\right\} \tag{5.2}$$

where the bases u_m in $h^{(J)}$ has been characterised in Section 2 and $\bar{v}_m^{(J,U)}$ is an orthonormal basis in $\bar{h}^{(J,U)}$ of eigenstates of $\bar{S}^2 = \bar{S}_x^2 + \bar{S}_y^2 + \bar{S}_z^2$ and $\bar{S}_z$ such that

$$(\bar{S}_x + i\bar{S}_y)\, \bar{v}_{m-1}^{(J,U)} = \sqrt{[(j+m)(j-m+1)]}\, \bar{v}_m^{(J,U)}$$

$\bar{S}_x$, $\bar{S}_y$, $\bar{S}_z$ being the generators of the representation $\bar{D}(R)$ of $SU(2)$ in $\bar{h}_0$.

In the case of an unstable particle with mass M, spin j, internal energy U, decay constant γ and positive (negative) intrinsic parity a generalisation of the forementioned embedding can be given if the following situation occurs:

(a) There exist $2j+1$ holomorphic vector-valued functions $\bar{v}_m^{(J)}(z) \in \bar{h}_0$ of the complex variable z, for z belonging to a suitable domain D, which are (i) eigenstates of $\bar{S}_z$ with eigenvalue m spanning for each

$z \in D$ a representation $\mathscr{D}^{(j)}$ of $SU(2)$, (ii) eigenstates of $\bar{I}_0$ all with the same eigenvalue $+1(-1)$. More specifically we assume that

$$\bar{S}_z \bar{v}_m^{(j)}(z) = m\bar{v}_m^{(j)}(z), \qquad m = -j, -j+1, \ldots, +j$$

$$\bar{D}(R)\,\bar{v}_m^{(j)}(z) = \sum_{m'=-j}^{+j} D_{m'm}^{(j)}(R)\,\bar{v}_{m'}^{(j)}(z), \qquad z \in D \tag{5.3}$$

with $D_{m'm}^{(j)} = (u_{m'}, D^{(j)} u_m)$, so that by (2.11)

$$\bar{S}_+ \bar{v}_{m-1}^{(j)}(z) = \sqrt{[(j+m)(j-m+1)]}\,\bar{v}_m^{(j)}(z) \tag{5.4}$$

$$\bar{I}_0 \bar{v}_m^{(j)}(z) = \bar{v}_m^{(j)}(z), \qquad (\bar{I}_0 \bar{v}_m^{(j)}(z) = -\bar{v}_m^{(j)}(z));$$

$$m = -j, \ldots, j-1, j \tag{5.3'}$$

(b) the expression

$$(\bar{T}_0(-1)^m \bar{v}_{-m}^{(j)}(z_1), \mathrm{e}^{-i\bar{E}_0 t}\,\bar{v}_m^{(j)}(z_2))_{\bar{h}_0} \overset{\text{def}}{=} F^{(j)}(z_1, z_2, t)$$

$$m = -j, -j+1, \ldots, +j \tag{5.5}$$

can be analytically continued in z_1 and z_2 to reach the point $\lambda = U - (i/2)\gamma$ for all $t \geqslant 0$ and finally

(c)

$$F^{(j)}(\lambda, \lambda, t) = Z_j^{-1}\,\mathrm{e}^{-i\lambda t}, \qquad t \geqslant 0 \tag{5.6}$$

Where

$$Z_j^{-1} = (\bar{T}_0(-1)^m \bar{v}_{-m}^{(j)}(z_1), \bar{v}_m^{(j)}(z_2))_{\bar{h}_0}|_{z_1 = z_2 = \lambda} \tag{5.7}$$

(5.5) as well as (5.7) does not depend on m, as one easily verifies taking into account equation (5.4).

We stress that in the weak embedding we shall perform, the functions $\bar{v}_m^{(j)}(z)$ play a role analogous to that of the common eigenstates $v_m^{(j,U)}$ of $\bar{S}_z$ and $\bar{E}_0$ in the strong embedding (5.2). Such a role of 'weak eigenstates' of $\bar{E}_0$ with eigenvalue λ appears clearly from property (c).

Now let us define a linear operator-valued function $\overleftarrow{G}_+(z)$ from $\mathscr{H}$ to $\bar{\mathscr{H}}$, holomorphic in z for $z \in D$, by the relation

$$\overleftarrow{G}_+(z)\left\{\sum_{m=-j}^{+j} f_m(\mathbf{p})\,u_m\right\} = \left\{\sum_{m=-j}^{+j} f_m(\mathbf{p})\,\bar{v}_m^{(j)}(z)\right\} Z_j^{1/2} \tag{5.8}$$

and the operator-valued function $\overleftarrow{G}_-(z)$ by

$$\overleftarrow{G}_-(z) = \bar{T}\overleftarrow{G}_+(z)\,T \tag{5.9}$$

or explicitly

$$\overleftarrow{G}_-(z)\left\{\sum_{m=-1}^{+j} f_m(\mathbf{p})\,u_m\right\} = \left\{\sum_{m=-j}^{+j} (-1)^m f_m(\mathbf{p})\,\bar{T}_0\,\bar{v}_{-m}^{(j)}(z)\right\} Z_j^{*1/2} \tag{5.10}$$

One has easily by (5.6), (5.8) and (5.10)

$$(\overleftarrow{G}_-(z_1) f, \mathrm{e}^{-i\bar{H}t}\,\overleftarrow{G}_+(z_2) g)_{\bar{\mathscr{H}}}|_{z_1 = z_2 = \lambda} = (f, V(t)\,g)_{\mathscr{H}} \tag{5.11}$$

In fact

$$(\overleftarrow{G}_-(z_1)f, e^{-i\bar{H}t}\overleftarrow{G}_+(z_2)g)_{\bar{\mathscr{H}}}\Big|_{\substack{z_1=\lambda\\ z_2=\lambda}} = \int d\mathbf{p}\, e^{-i(p^2/2M)t} \sum_{m,m'=-j}^{+j} f_m{}^*(\mathbf{p}) g_{m'}(\mathbf{p})$$

$$(-1)^m(\bar{T}_0\, \bar{v}_{-m}^{(j)}(z_1), e^{-iE_0t}\, \bar{v}_{m'}^{(j)}(z_2))_{\bar{h}_0}\Big|_{\substack{z_1=\lambda\\ z_2=\lambda}} Z_j$$

$$= \int d\mathbf{p} \exp(-ip^2/2Mt - i\lambda t) \sum_{m=-j}^{+j} f_m{}^*(\mathbf{p})\, g_m(\mathbf{p}) = (f, V(t)g)_{\mathscr{H}}$$

where it has been taken into account that

$$(\bar{T}_0\, \bar{v}_{-m}^{(j)}(z_1), e^{-iE_0t}\, \bar{v}_{m'}^{(j)}(z_2))_{\bar{h}_0}\big|_{z_1=z_2=\lambda} = 0 \qquad \text{for } m \neq m'$$

by analytic continuation from D, since

$$\bar{S}_z \bar{T}_0\, \bar{v}_{-m}^{(j)}(z_1) = m\bar{T}_0\, \bar{v}_m^{(j)}(z_1)$$
$$\bar{S}_z\, e^{-iE_0t}\, \bar{v}_{m'}^{(j)}(z_2) = m'\, e^{-iE_0t}\, \bar{v}_{m'}^{(j)}(z_2)$$

Furthermore one has easily

$$\bar{U}(g)\overleftarrow{G}_\pm(z) = \overleftarrow{G}_\pm(z)\, U(g), \qquad g \in \mathscr{G}_0, \qquad z \in D$$
$$\bar{P}\overleftarrow{G}_\pm(z) = \overleftarrow{G}_\pm(z) P, \qquad z \in D \tag{5.12}$$

Equation (5.12) is trivial for space inversion and for $g = (0, \mathbf{a}, \mathbf{v}, I)$; for $g = (0, \mathbf{0}, \mathbf{0}, R)$ one has

$$\overleftarrow{G}_+(z)\, O(R) \left\{ \sum_{m=-j}^{+j} f_m(\mathbf{p})\, u_m \right\} = \overleftarrow{G}_+(z) \left\{ \sum_{m=-j}^{+j} f_m(R^{-1}\mathbf{p})\, \bar{D}(R)\, u_m^{(j)} \right\}$$

$$= Z_j^{1/2} \left\{ \sum_{m,m'=-j}^{+j} f_m(R^{-1}\mathbf{p})\, \bar{v}_{m'}^{(j)}(z)\, (u_{m'}, D^{(j)}(R)\, u_m) \right\} = \bar{O}(R)$$

$$\cdot \left\{ \sum_{m=-j}^{+j} f_m(\mathbf{p})\, \bar{v}_m^{(j)}(z) \right\} Z_j^{1/2} = \bar{O}(R)\, \overleftarrow{G}_+(z) \left\{ \sum_{m=-j}^{+j} f_m(\mathbf{p})\, u_m \right\}$$

where equation (5.3) has been taken into account. A similar proof can be given for $\overleftarrow{G}_-(z)$. If the analytic vectors $\bar{v}_m^{(j)}(z)$ were continuable to the point λ equations (5.8), (5.11), (5.12) and (5.9) would define a 'strong' embedding with $\overleftarrow{G}_\pm = \overleftarrow{G}_\pm(\lambda)$. However, for $\gamma \neq 0$ one does not expect such a situation to occur: one expects that $\bar{v}_m^{(j)}(z)$ and therefore also $\overleftarrow{G}_+(z)$ are not continuable to λ, even if expression (5.5) is continuable. Let us assume for simplicity that $Z_j F^{(j)}(z_1, z_2, t)$ can be continued in z_1 and z_2 to λ, starting from a point $z_0 \in D$ by only one power expansion, i.e.

$$Z_j F^{(j)}(\lambda, \lambda, t)$$
$$= \sum_{r,s=0}^{\infty} \frac{(\lambda - z_0)^r}{r!} \frac{(\lambda - z_0)^s}{s!} \frac{\partial^r}{\partial z_1{}^r} \frac{\partial^s}{\partial z_2{}^s} Z_j F^{(j)}(z_1, z_2, t)\Big|_{\substack{z_1=z_0\\ z_2=z_0}} \qquad t \geqslant 0 \tag{5.13}$$

For an arbitrary $\varepsilon > 0$, one can choose n large enough, so that for a finite set τ of fixed time points $t_1, t_2, \ldots, t_k$ one has

$$\left| Z_j F^{(J)}(\lambda, \lambda, t) - \sum_{r,s=0}^{n} \frac{(\lambda - z_0)^r}{r!} \frac{(\lambda - z_0)^s}{s!} \frac{\partial^r}{\partial z_1{}^r} \frac{\partial^s}{\partial z_2{}^s} Z_j F^{(J)}(z_1, z_2, t) \Big|_{\substack{z_1 = z_0 \\ z_2 = z_0}} \right| < \varepsilon, \qquad t \in \tau \quad (5.14)$$

Defining

$$\overleftarrow{G}_+ = \sum_{s=0}^{n} \frac{(\lambda - z_0)^s}{s!} \frac{d^s}{dz^s} \overleftarrow{G}_+(z) \Big|_{z=z_0} \quad (5{\cdot}15)$$

$$\overleftarrow{G}_- = \bar{T} \overleftarrow{G}_+ T \quad (5.16)$$

one has

$$\overleftarrow{G}_+ \left\{ \sum_{m=-j}^{+j} g_m(\mathbf{p})\, u_m \right\} = \left\{ \sum_{m=-j}^{+j} \sum_{s=0}^{n} g_m(\mathbf{p}) \frac{(\lambda - z_0)^s}{s!} \frac{d^s \bar{v}_m^{(J)}(z)}{dz^s} \Big|_{z=z_0} \right\} Z_j^{1/2} \quad (5.17)$$

$$\overleftarrow{G}_- \left\{ \sum_{m=-j}^{+j} f_m(\mathbf{p})\, u_m \right\} = \left\{ \sum_{m=-j}^{+j} \sum_{r=0}^{n} (-1)^m f_m(\mathbf{p}) \frac{(\lambda^* - z_0{}^*)^r}{r!} \bar{T}_0 \frac{d^r \bar{v}_{-m}^{(J)}(z)}{dz^r} \Big|_{z=z_0} \right\} Z_j^{*1/2} \quad (5.18)$$

so that

$$(\overleftarrow{G}_- f, e^{-i\bar{H}t} \overleftarrow{G}_+ g)_{\bar{\mathscr{H}}} = \int d\mathbf{p}\, e^{-i(p^2/2M)t} \sum_{m=-j}^{+j} \sum_{r=0}^{n} \sum_{s=0}^{n} \frac{(\lambda - z_0)^r}{r!} \frac{(\lambda - z_0)^s}{s!} \cdot \frac{\partial^r}{\partial z_1{}^r} \frac{\partial^s}{\partial z_2{}^s} F^{(J)}(z_1, z_2, t) \Big|_{z_1 = z_2 = z_0} Z_j f_m{}^*(\mathbf{p})\, g_m(\mathbf{p})$$

Then, taking into account (5.14) one has

$$\left| (\overleftarrow{G}_- f, e^{-i\bar{H}t} \overleftarrow{G}_+ g)_{\bar{\mathscr{H}}} - \int d\mathbf{p}\, e^{-i(p^2/2M)t} Z_j \sum_{m=-j}^{+j} F^{(J)}(\lambda, \lambda, t) f_m{}^*(\mathbf{p})\, g_m(\mathbf{p}) \right|$$

$$\leqslant \varepsilon \int d\mathbf{p} \sum_{m=-j}^{+j} |f_m{}^*(\mathbf{p})|\, |g_m(\mathbf{p})| \leqslant \varepsilon \|f\|_{\mathscr{H}} \cdot \|g\|_{\mathscr{H}} = \varepsilon, \qquad t \in \tau,$$

$$f, g \in \mathscr{H}, \qquad \|f\|_{\mathscr{H}} = \|g\|_{\mathscr{H}} = 1 \quad (5.19)$$

and finally by (5.6) we have

$$|(\overleftarrow{G}_- f, e^{-i\bar{H}t} \overleftarrow{G}_+ g)_{\bar{\mathscr{H}}} - (f, V(t)\, g)_{\mathscr{H}}| \leqslant \varepsilon, \qquad t \in \tau,$$

$$f, g \in \mathscr{H}, \qquad \|f\|_{\mathscr{H}} = \|g\|_{\mathscr{H}} = 1 \quad (5.20)$$

By equations (5.12) and (5.15) one has immediately that

$$\bar{U}(g) \overleftarrow{G}_\pm = \overleftarrow{G}_\pm U(g), \qquad g \in \mathscr{G}_0$$

$$\bar{P} \overleftarrow{G}_\pm = \overleftarrow{G}_\pm P \quad (5.21)$$

Equations (5.20), (5.21) and (5.16) establish an embedding for the particle characterised by M, U, j, and (5.15) provides an explicit construction of the embedding operator $\overleftarrow{G}_+$. In the general case, in which the analytic continuation of expressions (5.5) to λ requires more than one power expansion, an obvious generalisation of this procedure leads essentially to the same result, the only difference being in more complicated coefficients in the expression (5.15) of $\overleftarrow{G}_+$ in terms of $(d^s/dz^s)\overleftarrow{G}_+(z)|_{z=z_0}$. If the series (5.13) by which the analytic continuation is given is uniformly convergent with respect to t, for t belonging to certain intervals δ_i of the positive time axes, the intervals δ_i can be included into the set τ. Now the problem arises of finding concretely vector-valued functions $\bar{v}_m^{(j)}(z)$ having the properties (a)–(c). In the case of a stable particle $\bar{v}_m^{(j)}(z)$ are z-independent eigenstates of $\bar{E}_0$ corresponding to the eigenvalue U. Such an eigenvalue is a pole of the resolvent $[\bar{E}_0 - z]^{-1}$; since $\bar{E}_0$ is self-adjoint such poles can only be found on the real axes.

The resolvent has first-order poles corresponding to the internal energies of the stable particles of mass M described by the field theory and a cut corresponding to the internal energies of two or more particle systems with total mass M. The resolvent cannot be analytically continued across the cut. However, by a suitable choice of an orthogonal projection operator $\bar{P}_0$ onto a subspace $\bar{K}$ of $\bar{h}_0$ it may quite well happen that the reduced resolvent

$$\bar{P}_0 \frac{1}{\bar{E}_0 - z} \bar{P}_0 \tag{5.22}$$

can be analytically continued from the upper to the lower half-plane across the cut, so that $\lambda = U - i(\gamma/2)$ is a pole of the continued operator (5.22). More specifically we shall assume that such a $\bar{P}_0$ exists and projects onto a subspace invariant under rotations, space and time reflections, i.e.

$$[\bar{P}_0, \bar{D}(R)] = 0, \qquad R \in SU(2)$$
$$[\bar{P}_0, \bar{I}_0] = 0, \qquad [\bar{P}_0, \bar{T}_0] = 0 \tag{5.23}$$

We take over at this point the formalism of reduced descriptions, which has been much employed in non-equilibrium statistical mechanics (see, for example, Lanz *et al.*, 1971, and Lanz & Lugiato, 1969). The following representation holds true:

$$\frac{1}{z - \bar{E}_0} = \Big\{ [\bar{P}_0 - \bar{N}_0(z)] \frac{1}{z + \bar{M}_0(z)} [\bar{P}_0 - \bar{N}_0{}^+(z^*)]$$
$$+ (1 - \bar{P}_0) \frac{1}{z - (1 - \bar{P}_0)\bar{E}_0(1 - \bar{P}_0)} (1 - \bar{P}_0) \Big\}, \qquad \operatorname{Im} z \neq 0 \tag{5.24}$$

where

$$\bar{M}_0(z) = \bar{P}_0\,\bar{E}_0(1-\bar{P}_0)\frac{1}{(1-\bar{P}_0)\,\bar{E}_0(1-\bar{P}_0)-z}(1-\bar{P}_0)\,\bar{E}_0\bar{P}_0 - \bar{P}_0\,\bar{E}_0\bar{P}_0 \tag{5.25}$$

$$\bar{N}_0(z) = (1-\bar{P}_0)\frac{1}{(1-\bar{P}_0)\,\bar{E}_0(1-\bar{P}_0)-z}(1-\bar{P}_0)\,\bar{E}_0\bar{P}_0 \tag{5.26}$$

The operator $\bar{M}_0(z)$ has the same role of $-\bar{E}_0$ in the expression of the reduced resolvent (5.22); in fact one has by (5.24)

$$\bar{P}_0\frac{1}{z-\bar{E}_0}\bar{P}_0 = \frac{1}{z+\bar{M}_0(z)}\bar{P}_0$$

but, differently from $\bar{E}_0$, it maps $\tilde{K}$ into itself and depends on z. By (5.25) and (5.26) the following relevant relation holds

$$\bar{N}_0{}^+(z_1{}^*)\,\bar{N}_0(z_2) = \frac{\bar{M}_0(z_1)-\bar{M}_0(z_2)}{z_1-z_2}, \qquad \operatorname{Im} z_1, \qquad \operatorname{Im} z_2 \neq 0 \tag{5.27}$$

the operator $\bar{N}_0(z)$ is holomorphic in z for $\operatorname{Im} z \neq 0$ and by (5.27) also $\bar{M}_0(z)-\bar{M}_0(z_0)$, $\operatorname{Im} z_0 \neq 0$, is holomorphic in z for $\operatorname{Im} z \neq 0$. Our basic assumptions about the subspace $\tilde{K}$ are as follows:

(i) $\bar{M}_0(z)$ can be analytically continued from the upper half-plane to yield an operator $\bar{M}_{0+}(z)$ holomorphic in a region D_+ containing the point λ (strictly one defines holomorphicity for bounded operators, but in our case we have $\bar{M}_0(z) = \bar{M}_0(z_0) + [\bar{M}_0(z) - \bar{M}_0(z_0)]$, where the closure of $\bar{M}_0(z) - \bar{M}_0(z_0)$ is bounded (Lanz *et al.*, 1971) and $\bar{M}_0(z_0)$ is independent of z, so that one considers precisely the analytic continuation of the closure of $\bar{M}_0(z) - \bar{M}_0(z_0)$).

(ii) The Kernel K of $\bar{M}_{0+}(\lambda) + \lambda$ in $\tilde{K}$ is not empty. Such a situation can occur only when $\operatorname{Im}\lambda \leqslant 0$ (Lanz *et al.*, 1971). Since $[\bar{D}(R), \bar{M}_{0+}(z)] = 0$, the Kernel K is invariant under rotations.

(iii) K contains a subspace $K^{(J)}$ invariant under $\bar{D}(R)$, $R \in SU(2)$, such that the restriction of $\bar{D}(R)$ to $K^{(J)}$ gives a representation $D^{(J)}$. Let us indicate by $\bar{u}_m^{(J)}$ the corresponding normalised eigenstates of $\bar{S}_z$ with the usual choice of phases.

(iv)
$$\bar{I}_0\,\bar{u}_m^{(J)} = \bar{u}_m^{(J)}\ (\bar{I}_0\,\bar{u}_m^{(J)} = -\bar{u}_m^{(J)}), \qquad m = -j, \ldots, j-1, j$$

Then the vector-valued functions

$$\bar{v}_m^{(J)}(z) = [\bar{P}_0 - \bar{N}_0(z)]\,u_m^{(J)}, \qquad m = -j, \ldots, j-1, j \tag{5.28}$$

are holomorphic in $D = \{z: \operatorname{Im} z > 0\}$ and have the properties (a), (b), (c). In fact (a) and equation (5.3) hold trivially since

$$[\bar{D}(R), \bar{N}_0(z)] = 0, \qquad [\bar{I}_0, \bar{N}_0(z)] = 0 \tag{5.29}$$

which follows from (5.23). To verify (b) and (c) let us calculate explicitly

$$F^{(j)}(z_1, z_2, t) = (-1)^m (\bar{T}_0 \bar{v}^{(j)}_{-m}(z_1), \mathrm{e}^{-i\bar{E}_0 t} \bar{v}^{(j)}_m(z_2))_{\bar{h}_0}$$

$$= -\frac{1}{2\pi i} \int\limits_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} dz \left([\bar{P}_0 - \bar{N}_0(z_1{}^*)]\, \bar{u}'^{(j)}_m, \frac{\exp(-izt)}{z - \bar{E}_0} [\bar{P}_0 - \bar{N}_0(z_2)]\, u^{(j)}_m \right)_{\bar{h}_0},$$

$t \geqslant 0$

where

$$\bar{u}'^{(j)}_m = (-1)^m \bar{T}_0 \bar{u}^{(j)}_{-m}$$

Taking into account equations (5.24), (5.27) and the first resolvent identity for $[(1 - \bar{P}_0)\bar{E}_0(1 - \bar{P}_0) - z]^{-1}$ we easily get

$$F^{(j)}(z_1, z_2, t) = -\frac{1}{2\pi i} \int\limits_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} dz\, \mathrm{e}^{-izt} \left(\bar{u}'^{(j)}_m, \left[\frac{1}{z - z_2} \left(1 + \frac{\bar{M}_0(z_1) - \bar{M}_0(z_2)}{z_1 - z_2} \right) \right.\right.$$
$$\left.\left. - \frac{z_1 + \bar{M}_0(z_1) - z - \bar{M}_0(z)}{z_1 - z} \frac{1}{z + \bar{M}_0(z)} \frac{z_2 + \bar{M}_0(z_2)}{z - z_2} \right] \bar{u}_m \right)_{\bar{h}_0'}, \qquad t \geqslant 0 \qquad (5.30)$$

It is important to notice that in such an expression the non-continuable operator $\bar{N}_0(z)$ (Lanz *et al.*, 1971) is eliminated and only the operator $\bar{M}_0(z)$, which is continuable by assumption, appears. Property (b) follows simply putting into (5.30) the continued operator $\bar{M}_{0+}(z_{1,2})$ in place of $\bar{M}_0(z_{1,2})$. Further, taking into account that by definition of $\bar{u}^{(j)}_m$ one has

$$[\bar{M}_{0+}(\lambda) + \lambda]\, \bar{u}^{(j)}_m = 0 \qquad (5.31)$$

we get

$$F^{(j)}_m(z_1, \lambda, t) = -\frac{1}{2\pi i} \int\limits_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} dz\, \mathrm{e}^{-izt} \left(\bar{u}'^{(j)}_m, \frac{1}{z - \lambda} \left[1 + \frac{\bar{M}_{0+}(z_1) + \lambda}{z_1 - \lambda} \right] \bar{u}^{(j)}_m \right)_{\bar{h}_0}$$

$$= \mathrm{e}^{-i\lambda t} \left(\bar{u}'^{(j)}_m, \left[1 + \frac{\bar{M}_{0+}(z_1) + \lambda}{z_1 - \lambda} \right] \bar{u}^{(j)}_m \right)_{\bar{h}_0} \qquad (5.32)$$

finally property (c) is obtained as follows

$$F^{(j)}_m(\lambda, \lambda, t) = \mathrm{e}^{-i\lambda t} \left(\bar{u}'^{(j)}_m, \left[1 + \left. \frac{d\bar{M}_{0+}(z)}{dz} \right|_{z=\lambda} \right] \bar{u}^{(j)}_m \right)_{\bar{h}_0}$$
$$= Z_j^{-1} \exp(-i\lambda t), \qquad t \geqslant 0 \qquad (5.33)$$

where

$$Z_j^{-1} = \left(\bar{u}'^{(j)}_m, \left[1 + \left. \frac{d\bar{M}_{0+}(z)}{dz} \right|_{z=\lambda} \right] \bar{u}^{(j)}_m \right)_{\bar{h}_0} \qquad (5.34)$$

As we have already stressed, the states $\bar{v}^{(j)}_m(z)$ are in some sense a generalisation of the common eigenstates of $\bar{E}_0$ and $\bar{S}_z$. In particular one expects that in the case of a stable particle with internal energy $\lambda = U$, the state $\bar{v}_m^{(j)}(U)$ built by the forementioned procedure is an eigenstate of $\bar{E}_0$ with eigenvalue U (Grecos, 1971). In fact, let us assume that

$$[\bar{M}_0(U) + U]\,\bar{u}_m^{(j)} = 0 \tag{5.35}$$

and that, indicating

$$\begin{gathered}[\bar{P}_0 - \bar{N}_0(z)]\,\bar{u}_m^{(j)} = \bar{v}_m^{(j)}(z)\\ s\text{-}\lim_{z\to U} \bar{v}_m^{(j)}(z) = \bar{v}_m^{(j)}(U)\end{gathered} \tag{5.36}$$

then by the identity

$$\frac{1}{z-\bar{E}_0}\,\bar{v}_m^{(j)}(z') = -\frac{1}{z-z'}[\bar{P}_0 - \bar{N}_0(z)]\frac{1}{z+\bar{M}_0(z)}[z' + \bar{M}_0(z')]\,\bar{u}_m^{(j)} + \frac{\bar{v}_m^{(j)}(z')}{z-z'}$$

which follows easily from (5.24), one has

$$s-\lim_{z'\to U}\frac{1}{z-\bar{E}_0}\,\bar{v}_m^{(j)}(z') = \frac{\bar{v}_m^{(j)}(U)}{z-U}$$

and since $[z-\bar{E}_0]^{-1}$ is bounded

$$\frac{1}{z-\bar{E}_0}\,\bar{v}_m^{(j)}(U) = \frac{1}{z-U}\,\bar{v}_m^{(j)}(U) \tag{5.37}$$

Since

$$\begin{aligned}\|\bar{v}_m^{(j)}(U)\|^2_{\bar{h}_0} &= \lim_{z\to U}([\bar{P}_0 - \bar{N}_0(z^*)]\,\bar{u}_m^{(j)}, [\bar{P}_0 - \bar{N}_0(z)]\,\bar{u}_m^{(j)})_{\bar{h}_0}\\ &= \left(\bar{u}_m^{(j)}, \left[1 + \left.\frac{d\bar{M}_0(z)}{dz}\right|_{z=U}\right]\bar{u}_m^{(j)}\right)_{\bar{h}_0} = Z_j^{-1}\end{aligned}$$

Z_j is a normalisation constant. Conversely, if $\bar{v}_m^{(j)}$ is an eigenstate of $\bar{E}_0$ with eigenvalue U and if the projector $\bar{P}_0$ is such that a neighbourhood I_U of $z = U$ and a $K > 0$ exist such that

$$\|[\bar{P}_0 - \bar{N}_0{}^+(z^*)]\,\bar{v}_m^{(j)}\|_{\bar{h}_0} < K, \qquad z \in I_U \tag{5.38}$$

then by the identity

$$[z + \bar{M}_0(z)]\,\bar{P}_0\,\bar{v}_m^{(j)} = (z-U)\,[\bar{P}_0 - \bar{N}^+(z^*)]\,v_m^{(j)} \tag{5.39}$$

which follows immediately applying (5.24) to the eigenstate $\bar{v}_m^{(j)}$ of $\bar{E}_0$ and multiplying at the left by $\bar{P}_0$, one has

$$[U + \bar{M}_0(U)]\,\bar{P}_0\,\bar{v}_m^{(j)} = 0 \tag{5.40}$$

Applying (5.24) to $\bar{v}_m^{(j)}$, multiplying at the left by $(1-\bar{P}_0)$ and taking (5.39) into account one gets the identity

$$\begin{aligned}(1-\bar{P}_0)\,\bar{v}_m^{(j)} &= -\bar{N}_0(z)\,\bar{P}_0\,\bar{v}_m^{(j)}\\ &\quad + (z-U)(1-\bar{P}_0)\frac{1}{z-(1-\bar{P}_0)\,\bar{E}_0(1-\bar{P}_0)}(1-\bar{P}_0)\,\bar{v}_m^{(j)}\end{aligned} \tag{5.41}$$

Finally if one assumes that (i) a neighbourhood I_U' of $z = U$ and a $K' > 0$ exist such that

$$\left\| (1 - \bar{P}_0) \frac{1}{z - (1 - \bar{P}_0)\bar{E}_0(1 - \bar{P}_0)} (1 - \bar{P}_0)\bar{v}_m^{(J)} \right\|_{\bar{h}_0} < K', \qquad z \in I_U' \quad (5.42)$$

and that (ii) $s\text{-}\lim_{z \to U} \bar{v}_m^{(J)}(z)$ exists, where $\bar{v}_m^{(J)}(z) = [\bar{P}_0 - \bar{N}_0(z)]\bar{P}_0\bar{v}_m^{(J)}$, one concludes from (5.41) that

$$\bar{v}_m^{(J)} = s\text{-}\lim_{z \to U} \bar{v}_m^{(J)}(z) \qquad (5.43)$$

6. *An Example*

Let us illustrate briefly the embedding technique in the framework of the 'Galilee' model (Levy-Leblond, 1967). We recall the Hamiltonian of such a model: $\bar{H} = \bar{H}_F + \bar{H}_I$,

$$\bar{H}_F = \int d\mathbf{p}\left(\frac{p^2}{2\mathscr{M}} + U_0\right)\psi_V{}^+(\mathbf{p})\,\psi_V(\mathbf{p}) + \int d\mathbf{p}'\frac{p'^2}{2M}\psi_N{}^+(\mathbf{p}')\,\psi_N(\mathbf{p}')$$
$$+ \int d\mathrm{k}\frac{\mathrm{k}^2}{2m}a^+(\mathrm{k})\,a(\mathrm{k})$$
$$\bar{H}_I = \lambda_0 \int d\mathbf{p}\,d\mathbf{q}\,f(\omega)\left[\psi_V{}^+(\mathbf{p})\,\psi_N\left(\frac{M}{\mathscr{M}}\mathbf{p} + \mathbf{q}\right)a\left(\frac{m}{\mathscr{M}}\mathbf{p} - \mathbf{q}\right) + h.c\right],$$
$$\mathscr{M} = M + m, \qquad \omega = \frac{q^2}{2\mu}, \qquad \mu = \frac{mM}{\mathscr{M}} \qquad (6.1)$$

U_0 is the internal energy of the bare V-quanta; $\psi_V(\mathbf{p})$, $\psi_N(\mathbf{p}')$ are spinless fermion fields, $a(\mathrm{k})$ a spin zero boson field which obey the canonical commutation and anti-commutation relations; $f(\omega)$ is a cutoff function; we shall not consider in this paper the local coupling limit $f(\omega) \to 1$.

To describe the embedding of 'V-type' (stable or unstable) particles we consider the projector onto the eigenspace of the operator $\bar{E}_{0F}$ with eigenvalue U_0, where $\bar{E}_{0F}$ is the internal energy operator for the free particles. One obtains

$$\bar{M}_0^{(V)}(z) = \mathscr{M}^{(V)}(z)\,\bar{P}_0$$
$$\mathscr{M}^{(V)}(z) = -U_0 + \lambda_0{}^2 \int d\mathbf{q}\frac{f^2(\omega)}{\omega - z} \qquad (6.2)$$

$\bar{M}_0^{(V)}(z)$ is analytically continuable into the lower half-plane provided $f(\omega)$ is analytic. To each solution of equation $\mathscr{M}_+^{(V)}(\bar{z}) + \bar{z} = 0$ corresponds a particle with internal energy U and mean lifetime τ, such that

$$U = \mathrm{Re}\,\bar{z}, \qquad \tau = -[2\,\mathrm{Im}\,\bar{z}]^{-1} \qquad (6.3)$$

One immediately verifies that in the case of stable V particle equation $\mathscr{M}^{(V)}(\bar{z}) + \bar{z} = 0$ coincides with the equations given in Levy-Leblond (1967) and that equation (5.36) gives the dressed V-particle state. One has further that $[\mathscr{M}^{(V)}(z)+z]^{-1}$ is the full V-particle propagator, so that one sees from (6.3) that the embedding prescription for U and τ coincides with the non-relativistic version of the prescription for unstable particles given by Peierls (1956). We stress that each solution of equation $\mathscr{M}_{+}^{(V)}(\bar{z})+\bar{z}=0$ is to our opinion a candidate to correspond to an unstable particle. Therefore in our theory many particles can correspond to the same field. The same situation is met, in a rather different context, in the treatment of relativistic unstable particle of Matthews and Salam (1958).

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